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Logarithms

$$\boxed{\log = \text{exponent}}$$

$\log_B A = C$ (log form)

$A = B^C$ (expontential form)

Examples

$\log_{10} 100 = 2$

$100 = 10^2$

$\log_2 8 = 3$

$8 = 2^3$

It is very important to be able to change for log form to exponential form and vice versa.

Laws of logarithms

As logs are indices, the laws of logs are directly related to the laws of indices.

	Laws of logs	Numerical example
1.	$\log_a(xy) = \log_a x + \log_a y$	$\log_2 32 = \log_2(4 \times 8) = \log_2 4 + \log_2 8 = 2 + 3 = 5$
2.	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_3 27 = \log_3\left(\frac{81}{3}\right) = \log_3 81 - \log_3 3 = 4 - 1 = 3$
3.	$\log_a(x^q) = q \log_a x$	$\log_2 8 = \log_2 2^3 = 3 \log_2 2 = 3(1) = 3$
4.	$\log_a 1 = 0$	$1 = 5^0$, thus $\log_5 1 = 0$
5.	$\log_a\left(\frac{1}{x}\right) = -\log_a x$	$\log_2\left(\frac{1}{16}\right) = -\log_2 16 = -4$
6.	$\log_a a = 1$ $\log_a(a^x) = x$	$\log_4 4 = 1$ $\log_5(5^3) = 3$
7.	$a^{\log_a x} = x$	$6^{\log_6 4} = 4$
8.	$\log_b x = \frac{\log_a x}{\log_a b}$	$\log_4 64 = \frac{\log_2 64}{\log_2 4} = \frac{6}{2} = 3$

Note: Law 8 is usually referred to as the **change of base law**.

Exercise

1. Write each of the following in the form $a = b^c$.
(i) $\log_2 16 = 4$ (ii) $\log_3 81 = 4$ (iii) $\log_{10} 1{,}000 = 4$ (iv) $\log_5 125 = 3$
(v) $\log_6 36 = 2$ (vi) $\log_4 2 = \frac{1}{2}$ (vii) $\log_{27} 3 = \frac{1}{3}$ (viii) $\log_4 8 = \frac{3}{2}$
2. Write each of the following in the form $\log_b a = c$.
(i) $100 = 10^2$ (ii) $8 = 2^3$ (iii) $27 = 3^3$ (iv) $49 = 7^2$
(v) $4 = 16^{\frac{1}{2}}$ (vi) $9 = 27^{\frac{2}{3}}$ (vii) $4 = 4$ (viii) $1 = 8^0$
3. Evaluate each of the following.
(i) $\log_2 8$ (ii) $\log_4 16$ (iii) $\log_3 81$ (iv) $\log_5 125$
(v) $\log_{10} 10{,}000$ (vi) $\log_2 32$ (vii) $\log_5 5$ (viii) $\log_4 1$
(ix) $\log_2 \frac{1}{2}$ (x) $\log_7 \frac{1}{49}$ (xi) $\log_4 32$ (xii) $\log_{16} 8$
(xiii) $\log_9 27$ (xiv) $\frac{1}{3}\log_2 8$ (xv) $\log_{27} \frac{1}{3}$ (xvi) $\log_2 2\sqrt{2}$
4. Evaluate each of the following.
(i) $\log_a a^2$ (ii) $\log_a a^3$ (iii) $(\log_6 4 + \log_6 9)^2$
(iv) $(\log_5 25 + \log_5 15 - \log_5 3)^4$ (v) $\log 3 + \log 16 - \log 4 - \log 12$
5. (i) $f(x) = 2\log_5 x$. Evaluate: (a) $f(5)$ (b) $f(25)$ (c) $f(\frac{1}{5})$ (d) $f(\sqrt{5})$.
(ii) If $\log_4 x = 1 - p$ and $\log_4 y = 1 + p$, evaluate xy.
6. Evaluate $\log\frac{p}{q} + \log\frac{q}{r} + \log\frac{r}{p}$.
7. (i) Use the fact that $\log_b a = \dfrac{\log_c a}{\log_c b}$ to evaluate (a) $\log_{27} 81$ (b) $\log_{32} 8$.
(ii) Evaluate $(\log_b a)(\log_c b)(\log_a c)$.
(iii) Show that $\log_b a = \dfrac{1}{\log_a b}$.
(iv) If $x > 0$ and $x \neq 1$, show that $\dfrac{1}{\log_2 x} + \dfrac{1}{\log_3 x} + \dfrac{1}{\log_5 x} = \dfrac{1}{\log_{30} x}$.
8. If $\log_r p = \log_r 2 + 3\log_r q$, express p in terms of q.
9. If $\log_a y = 2\log_a x - \log_a 5$, express y in terms of x.
10. If $3\log_2 y = 3 + \log_2(x + 4)$, show that $y^3 = 8(x + 4)$.

Answers

1. (i) $16 = 2^4$ (ii) $81 = 3^4$ (iii) $1{,}000 = 10^4$ (iv) $125 = 5^3$ (v) $36 = 6^2$ (vi) $2 = 4^{\frac{1}{2}}$
(vii) $3 = 27^{\frac{1}{3}}$ (viii) $8 = 4^{\frac{3}{2}}$
2. (i) $\log_{10}100 = 2$ (ii) $\log_2 8 = 3$ (iii) $\log_3 27 = 3$ (iv) $\log_7 49 = 2$ (v) $\log_{16} 4 = \dfrac{1}{2}$
(vi) $\log_{27} 9 = \dfrac{2}{3}$ (vii) $\log_4 4 = 1$ (viii) $\log_8 1 = 0$
3. (i) 3 (ii) 2 (iii) 4 (iv) 3 (v) 4 (iv) 5 (vii) 1 (viii) 0 (ix) -1 (x) -2
(xi) $\dfrac{5}{2}$ (xii) $\dfrac{3}{4}$ (xiii) $\dfrac{3}{2}$ (xiv) 1 (xv) $-\dfrac{1}{3}$ (xvi) $\dfrac{3}{2}$

4. (i) 2 (ii) 3 (iii) 4 (iv) 81 (v) 0 5. (i) (a) 2 (b) 4 (c) -2 (d) 1 (ii) 16

6. 0 7. (i) (a) $\dfrac{4}{3}$ (b) $\dfrac{3}{5}$ (ii) 1 8. $p = 2q^3$ 9. $y = \dfrac{x^2}{5}$

Example

Evaluate :

(i) $\log_2 5 + \log_2 1.6$ (ii) $\log_4 28 - \log_4 7$

(iii) $2\log_{10} 5 + \log_{10} 4$

Solution :

(i) $\log_2 5 + \log_2 1.6$

$= \log_2 (5 \times 1.6)$

$= \log_2 8$

$= 3$

(ii) $\log_4 28 - \log 7$

$= \log_4 \left(\frac{28}{7}\right)$

$= \log_4 4$

$= 1$

(iii) $2\log_{10} 5 + \log_{10} 4$

$= \log_{10} 5^2 + \log_{10} 4$

$= \log_{10} 25 + \log_{10} 4$

$= \log_{10} (25 \times 4)$

$= \log_{10} 100$

$= 2$

Equations involving logs

Using the rules of logs, there are two methods for solving equations that contain logs.

Method 1: $\log_B A = \log_B C$

$\therefore \quad A = C$

One log on the left and one log on the right

Note: Bases must be the same.

Example: $\log_4(5x-3) = \log_4 7$

$\therefore \quad 5x - 3 = 7$

$5x = 10$

$x = 2$

Check: When $x = 2$, $\log_4(5x-3) = \log_4(10-3) = \log_4 7$

Method 2: $\log_B A = C$

$\therefore \quad A = B^C$

Go from log form to exponential form

Example: $\log_2(x^2-4) = 5$

$x^2 - 4 = 2^5$

$x^2 - 4 = 32$

$x^2 = 36$

$x = \pm\sqrt{36} = \pm 6$

Example

$\log_2 x + \log_2 (x-2) = \log_2 8$

solution :

$$\log_2 x + \log_2 (x-2) = \log_2 8$$

$$\log_2 x(x-2) = \log_2 8$$

$$x(x-2) = 8$$

$$x^2 - 2x = 8$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = 4 \quad \text{or} \quad x = -2$$

Reject $x = -2$, as substitution into the original equation gives $\log_2(-2)$ or $\log_2(-4)$ which are not defined.

Logs are only defined for positive numbers.

Check $x = 4$

$$\log_2 x + \log_2 (x-2) = \log_2 8$$

$$\log_2 4 + \log_2 (4-2) = 3$$

$$\log_2 4 + \log_2 2 = 3$$

$$2 + 1 = 3$$

$$3 = 3 \quad \checkmark$$

Example

Solve for x: $\log_4 x - \log_4(x-2) = \frac{1}{2}$

Solution:

$$\log_4 x - \log_4(x-2) = \frac{1}{2}$$
$$\log_4\left(\frac{x}{x-2}\right) = \frac{1}{2}$$
$$\frac{x}{x-2} = 4^{\frac{1}{2}}$$
$$\frac{x}{x-2} = 2$$
$$x = 2x-4$$
$$-x = -4$$
$$x = 4$$

Example

Solve for x: $\log_2(x^2-10) - \log_2 x = 2\log_2 3$

Solution:

$$\log_2(x^2-10) - \log_2 x = 2\log_2 3$$
$$\log_2\left(\frac{x^2-10}{x}\right) = \log_2 3^2$$
$$\frac{x^2-10}{x} = 9$$
$$x^2-10 = 9x$$
$$x^2-9x-10 = 0$$
$$(x+1)(x-10) = 0$$
$$x=-1 \text{ or } x = 10$$

$\log A - \log B = \log\left(\frac{A}{B}\right)$

$n\log A = \log A^n$

Reject $x=-1$ because logs are only defined for positive numbers.

$\therefore \log_2(-1)$ is not defined

$\therefore x = 10$

$(\log_2 3^2 = \log_2 9)$

Example

(i) Solve for x : $\log_2(x+6) - \log_2(x+2) = 1$.

(ii) Verify your answer.

Solution :

(i) $\log_2(x+6) - \log_2(x+2) = 1$

$$\log_2\left(\frac{x+6}{x+2}\right) = 1$$

$$\frac{x+6}{x+2} = 2^1$$

$$\frac{x+6}{x+2} = 2$$

$$x+6 = 2(x+2)$$

$$x+6 = 2x+4$$

$$x = 2$$

(ii) $\log_2(x+6) - \log(x+2) = \underline{1}$

$x=2$: $\log_2(2+6) - \log_2(2+2)$

$= \log_2 8 - \log_2 4$

$= 3 - 2 = 1$

$= 1$ $\therefore$ answer is correct

Example

Solve for x:

(i) $\log_a(2x+3) + \log_a(x-2) = 2\log_a(x+4)$, for $x > 2$.

(ii) Explain why $x > 2$.

Solution:

(i) $\log_a(2x+3) + \log_a(x-2) = 2\log_a(x+4)$

$$\log_a(2x+3)(x-2) = \log_a(x+4)^2$$

$$(2x+3)(x-2) = (x+4)^2$$

$$2x^2 - x - 6 = x^2 + 8x + 16$$

$$x^2 - 9x - 22 = 0$$

$$(x+2)(x-11) = 0$$

$$x = -2 \quad \text{OR} \quad x = 11$$

(ii) Logs are only defined for positive numbers

$2x + 3 > 0$	$x - 2 > 0$	$x + 4 > 0$
$2x > -3$	$x > 2$	$x > -4$
$x > -\frac{3}{2}$		

$\therefore \quad x > 2$

Example

Solve the simultaneous equations:

$2\log x = \log(x+y)$ and $\log y = \log 2 + \log(x-1)$,

where $x > 1$, $y > 0$.

Solution:

$$2\log x = \log(x+y)$$
$$\log x^2 = \log(x+y)$$
$$x^2 = x + y \quad (1)$$

$$\log y = \log 2 + \log(x-1)$$
$$\log y = \log 2(x-1)$$
$$y = 2(x-1)$$
$$y = 2x-2 \quad (2)$$

Now solve (1) and (2)

$$x^2 = x + y$$
$$x^2 = x + 2x - 2 \qquad (y = 2x-2 \text{ from } (2))$$
$$x^2 = 3x - 2$$
$$x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$
$$x = 1 \text{ or } x = 2 \qquad (\text{reject } x = 1, \text{ as } x > 1)$$

$x = 2$

$$y = 2x - 2$$
$$y = 2(2) - 2 = 4 - 2 = 2$$

$\therefore$ $x = 2$ and $y = 2$

Example

If $\log\left(\frac{a+b}{5}\right) = \frac{1}{2}(\log a + \log b)$,

show that $a^2 + b^2 = 23ab$.

Solution :

$$\log\left(\frac{a+b}{5}\right) = \frac{1}{2}(\log a + \log b)$$

$$2\log\left(\frac{a+b}{5}\right) = \log a + \log b$$

(multiply both sides by 2)

$$\log\left(\frac{a+b}{5}\right)^2 = \log ab$$

$$\therefore \left(\frac{a+b}{5}\right)^2 = ab$$

$$\frac{a^2+2ab+b^2}{25} = ab$$

$$a^2+2ab+b^2 = 25ab$$

$$a^2+b^2 = 23ab$$

Example

Given that $\log_a 2 = p$ and $\log_a 3 = q$, where $a > 0$, write each of the following in terms of p and q,

(i) $\log_3 \left(\frac{8}{3}\right)$ (ii) $\log_a \left(\frac{9a^2}{16}\right)$

Solution:

Given $\log_a 2 = p$ and $\log_a 3 = q$

(i)
$$\log_a \left(\frac{8}{3}\right)$$
$$= \log_a 8 - \log_a 3$$
$$= 3 \log_a 2 - \log_a 3$$
$$= 3p - q$$

$$\left(\begin{aligned} &\log_a 8 \\ &= \log_a 2^3 \\ &= 3 \log_a 2 \end{aligned} \right)$$

(ii)
$$\log_a \left(\frac{9a^2}{16}\right)$$
$$= \log_a 9a^2 - \log_a 16$$
$$= \log_a 9 + \log_a a^2 - \log_a 16$$
$$= \log_a 3^2 + \log_a a^2 - \log_a 2^4$$
$$= 2 \log_a 3 + 2 \log_a a - 4 \log_a 2 \quad (\log_a a = 1)$$
$$= 2q + 2 - 4p$$

Example

A sample of radioactive material decay can be modelled by the function:

$$D(t) = Ae^{bt}$$

where A and b are constants and t the time passed in weeks.
Initially, 50g of material is purchased and in 5 weeks has decayed to 25g.
Calculate, exactly, the value of A and the value of b.

Solution:

Initially, means $t = 0$ and 50g of material

Given: $D(0) = 50$

$$\therefore A e^{b(0)} = 50$$

$$A e^{0} = 50$$

$$A = 50$$

$(e^0 = 1)$

After 5 weeks means $t = 5$ and 25g of material

Given: $D(5) = 25$

$$\therefore 50 e^{b(5)} = 25$$

$$50e^{5b} = 25$$

$$e^{5b} = \frac{25}{50}$$

$$e^{5b} = \frac{1}{2}$$

$$\ln e^{5b} = \ln\left(\tfrac{1}{2}\right)$$

$$5b \ln e = \ln\left(\tfrac{1}{2}\right)$$

$$5b = \ln\left(\tfrac{1}{2}\right)$$

$$b = \tfrac{1}{5} \ln\left(\tfrac{1}{2}\right)$$

$$\text{or } b = -\tfrac{1}{5} \ln 2$$

$(\ln e = \log_e e = 1)$

$\left(\ln\left(\tfrac{1}{2}\right) = \ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2\right)$

Example

Solve for x : $\log_2(x-1) = \log_4(4x-7)$

Solution:

Conection between the bases, $4 = 2^2$

$\therefore$ change base 4 to base 2.

Rule: $\log_B A = \dfrac{\log_C A}{\log_C B}$

$$\therefore \log_4(4x-7) = \frac{\log_2(4x-7)}{\log_2 4} = \frac{\log_2(4x-7)}{2}$$

$$\log(x-1) = \frac{\log_2(4x-7)}{2}$$

$$2\log_2(x-1) = \log_2(4x-7)$$

$$\log_2(x-1)^2 = \log_2(4x-7)$$

$$\therefore (x-1)^2 = 4x-7$$

$$x^2-2x+1 = 4x-7$$

$$x^2-6x+8 = 0$$

$$(x-2)(x-4) = 0$$

$$x = 2 \text{ or } x = 4$$

Note:

It is usually easier to change to the smaller base.

Escample

Solve for x : $\log_4 x + 2\log_x 4 = 3$

Solution :

First write the equation in terms of one base only. Base x or base 4 It is usually easier to change the bases to the constant.

$\log_4 x$ has a base that is constant.

$$\log_x 4 = \frac{\log_4 4}{\log_4 x} = \frac{1}{\log_4 x}$$

$$\therefore\ 2\log_x 4 = 2\,\frac{1}{\log_4 x} = \frac{2}{\log_4 x}$$

$$\log_4 x + \frac{2}{\log_4 x} = 3$$

(let $\log_4 x = a$)

$$a + \frac{2}{a} = 3$$

$$a^2 + 2 = 3a$$

$$a^2 - 3a + 2 = 0$$

$$(a-1)(a-2) = 0$$

$$a = 1 \quad \text{or} \quad a = 2$$

$$\therefore\ \log_4 x = 1 \quad \text{or} \quad \log_4 x = 2$$

$$x = 4^1 \quad \text{or} \quad x = 4^2$$

$$x = 4 \quad \text{or} \quad x = 16$$

Example

Find the value of x which satisfies the equation $2e^{-x} - e^{x} + 1 = 0$.

(Hint : let $a = e^{x}$)

Solution :

$$e^{x} = a \quad \therefore \quad e^{-x} = \frac{1}{e^{x}} = \frac{1}{a}$$

$$\therefore \quad 2e^{-x} = 2\frac{1}{a} = \frac{2}{a}$$

$$2e^{-x} - e^{x} + 1 = 0$$

$$\frac{2}{a} - a + 1 = 0$$

$$2 - a^2 + a = 0$$

$$-a^2 + a + 2 = 0$$

$$a^2 - a - 2 = 0$$

$$(a+1)(a-2) = 0$$

$$a = -1 \text{ OR } a = 2$$

$$a = 2$$

$$e^{x} = 2$$

$$\ln e^{x} = \ln 2$$

$$x \ln e = \ln 2$$

$$x(1) = \ln 2$$

$$x = \ln 2$$

Note : $e^{x} > 0$

$\therefore \quad a = -1$

$e^{x} = -1$ has not solution

Solve for x:

$3e^{x} - 7 + 2e^{-x} = 0$

Solution:

Let $a = e^{x}$ (This is the substitution)

$3e^{x} = 3a$

$2e^{-x} = 2\left(\frac{1}{e^{x}}\right) = \frac{2}{e^{x}} = \frac{2}{a}$

$3e^{x} - 7 + 2e^{-x} = 0$

$3a - 7 + \frac{2}{a} = 0$

$3a^{2} - 7a + 2 = 0$

$(3a-1)(a-2) = 0$

$3a - 1 = 0$	$a - 2 = 0$
$a = \frac{1}{3}$	$a = 2$

Reverse the substitution!

$a = 2$	$a = \frac{1}{3}$
$e^{x} = 2$	$e^{x} = \frac{1}{3}$
$\ln e^{x} = \ln 2$	$\ln e^{x} = \ln\left(\frac{1}{3}\right)$
$x \ln e = \ln 2$	$x \ln e = \ln\left(\frac{1}{3}\right)$
$x(1) = \ln 2$	$x(1) = \ln\left(\frac{1}{3}\right)$
$x = \ln 2$	$x = \ln\left(\frac{1}{3}\right)$ or $-\ln 3$

Note: $\ln e = \log_{e} e = 1$

$\ln\left(\frac{1}{3}\right) = \ln 1 - \ln 3 = -\ln 3$

$(\ln 1 = 0)$